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Coupling Boundary Integral and Finite Element Methods for the Oseen Coupled Problem

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Abstract—In this paper, we represent an Oseen coupled problem and related numerical method for solving the nonstationary Navier-Stokes problem in an unbounded domain. The Oseen coupled problem consists of a coupling between the Navier-Stokes problem in an inner region and Oseen problem in an outer region. The related numerical method consists of coupling the boundary integral and the finite element method to solve the coupled problem. The variational formulation of the coupled problem and its well posedness are obtained. The optimal error estimates between the numerical solution of the Oseen coupled problem and the exact solution of the Navier-Stokes problem are provided.

Keywords—Boundary integral, Finite element, Navier-Stokes problem, Oseen problem.

1. INTRODUCTION

Let Ω_0 be a bounded domain in R^2 with smooth boundary Γ and let $\Omega = R^2\backslash(\Omega_0 \cup \Gamma)$. The nonstationary Navier-Stokes equations of the viscous incompressible flow in Ω are given as

$$\begin{aligned} &\frac{\partial u^*}{\partial t} - \nu\Delta u^* + (u^* \cdot \nabla)u^* + \nabla p^* = f^*, \quad \operatorname{div} u^* = 0, && \text{in } \Omega \times (0,T], \\ &u^*|_\Gamma = 0, \quad \lim_{|x|\to\infty} u^*(x,t) = w_0, && \forall t \in [0,T], \\ &u^*(x,0) = u_0^*(x), && \text{in } \Omega, \end{aligned} \tag{S}$$

where $u^* = u^*(x,t)$ is the velocity vector, $p^* = p^*(x,t)$ is the pressure, $\nu = \mathrm{Re}^{-1}$, $\mathrm{Re} > 0$ is the Reynolds number, u_0 is an initial velocity vector satisfying

$$\operatorname{div} u_0^* = 0, \quad \text{in } \Omega, \qquad u_0^* \cdot n|_\Gamma = 0, \qquad \lim_{|x|\to\infty} u_0^*(x) = w_0,$$

and $f^* = f^*(x,t)$ represents a prescribed body force with a compact support in Ω. Here $T > 0$ is a constant and $w_0 = (w_{01}, w_{02})$ is a constant vector.

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Let us now introduce a smooth artificial boundary $\Gamma_2 = \{x \in \Omega;\ |x| = R_0\}$ embedded in Ω, separating an unbounded part Ω_2 from a bounded part Ω_1 such that Ω_1 contains the support of f and $u^*(x,t) - w_0$ is sufficiently small in Ω_2. We shall denote by n the unit outward (from Ω_0) normal to Γ and (from Ω_2) normal to Γ_2.

By virtue of [1,2], there exists a vector function $\psi(x) \in H^2(\Omega)^2$ such that supp $\psi \subset \Omega_1$ and

$$\operatorname{div}\psi = 0, \quad \text{in } \Omega_1, \qquad \psi|_\Gamma = -w_0, \qquad \psi|_{\Gamma_2} = 0.$$

By setting

$$\bar u(x,t) = u^*(x,t) - w_0 - \psi(x), \qquad \bar p(x,t) = p^*(x,t),$$

in (S), we derive that $(\bar u, \bar p)$ satisfies

$$\begin{aligned}
&\frac{\partial \bar u}{\partial t} - \nu\Delta\bar u + (\bar u\cdot\nabla)\bar u + (w_0\cdot\nabla)\bar u + (\psi\cdot\nabla)\bar u + (\bar u\cdot\nabla)\psi + \nabla\bar p = f, \quad \operatorname{div}\bar u = 0, && \text{in } \Omega\times(0,T],\\
&\bar u(x,t)|_\Gamma = 0, \quad \lim_{|x|\to\infty}\bar u(x,t) = 0, && \forall\, t\in[0,T], \qquad \text{(S)}\\
&\bar u(x,0) = u_0(x), && \text{in } \Omega.
\end{aligned}$$

For reasons of computational economy and the fact that the inertial term $(u\cdot\nabla)u$ is small in Ω_2, we consider instead the Oseen coupled problem (S′) for approximating the Navier-Stokes problem (S):

$$\begin{aligned}
&\frac{\partial u}{\partial t} - \nu\Delta u + X_{\Omega_1}(u\cdot\nabla)u + (\psi\cdot\nabla)u + (u\cdot\nabla)\psi\\
&\quad + (w_0\cdot\nabla)\bar u + \nabla p = f, \quad \operatorname{div} u = 0, && \text{in } \Omega\times(0,T],\\
&u(x,t)|_\Gamma = 0, \quad \lim_{|x|\to\infty} u(x,t) = 0, && \forall\, t\in[0,T], \qquad \text{(S}')\\
&u(x,0) = u_0(x), && \text{in } \Omega,
\end{aligned}$$

where

$$X_{\Omega_1}(x) = \begin{cases} 1, & x \in \bar\Omega_1,\\ 0, & x \bar\in \bar\Omega_1.\end{cases}$$

In Section 5, we shall prove that the difference between $(\bar u, \bar p)$ and (u,p) is small in $\Omega\times[0,T]$, if R_0 is chosen such that Ω_1 contains the support of $f(x,t)$ and that

$$|\bar u(x,t)| \le \varepsilon, \quad |u(x,t)| \le \varepsilon, \qquad \forall\,(x,t) \in \bar\Omega_2\times[0,T], \tag{1.1}$$

for a fixed small number ε. For convenience, we take the equivalent form of problem (S′),

$$\begin{aligned}
\frac{\partial u}{\partial t} - \nabla\cdot\sigma(u,p) + (\psi\cdot\nabla)u + (u\cdot\nabla)\psi + (w_0\cdot\nabla)u + (u\cdot\nabla)u = f, \quad \operatorname{div} u = 0, &\qquad \text{in } \Omega_1\times(0,T],\\
u = 0, &\qquad \text{on } \Gamma\times[0,T],\\
\lambda^- = \lambda^+, &\qquad \text{on } \Gamma_2\times[0,T],\\
u(x,0) = u_0, &\qquad \text{in } \Omega_1,\\
\frac{\partial u}{\partial t} - \nabla\cdot\sigma(u,p) + (w_0\cdot\nabla)u = 0, \quad \operatorname{div} u = 0, &\qquad \text{in } \Omega_2\times(0,T], \qquad \text{(S}')\\
u^+ = u^-, &\qquad \text{on } \Gamma_2\times[0,T],\\
\lim_{|x|\to\infty} u(x,t) = 0, &\qquad \forall\, t\in[0,T],\\
u(x,0) = 0, &\qquad \text{in } \Omega_2,
\end{aligned}$$

where

$$u^- = \lim_{x\to\Gamma_2}(u|_{\Omega_1}), \qquad \lambda^- = \lim_{x\to\Gamma_2}(\sigma|_{\Omega_1})\cdot n,$$
$$u^+ = \lim_{x\to\Gamma_2}(u|_{\Omega_2}), \qquad \lambda^+ = \lim_{x\to\Gamma_2}(\sigma|_{\Omega_2})\cdot n,$$
$$\sigma(u,p) = (\sigma_{ij}(u,p)), \sigma_{ij}(u,p) = -\delta_{ij}p + 2\nu\mathcal{E}_{ij}(u), \mathcal{E}_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \qquad i,j = 1,2.$$

2. THE VARIATIONAL FORMULATION

In order to reduce the problem in Ω_2 to an integral equation on Γ_2, the fundamental solution (U_k, P_k) of the nonstationary Oseen equations

$$\begin{gathered}\frac{\partial U_k}{\partial t} - \nabla\cdot\sigma(U_k,P_k) + (w_0\cdot\nabla)U_k = \delta(x-y)\delta(t-\tau)e_k, \\ \operatorname{div} U_k = 0,\end{gathered} \tag{2.1}$$

will be employed. By the usual method (see [3]), we derive

$$P_k(x-y,t-\tau) = \frac{1}{2\pi}\frac{x_k-y_k}{|x-y|^2}\delta(t-\tau), \tag{2.2}$$

$$\begin{aligned}U_k(x-y,t-\tau) &= \frac{H(t-\tau)}{\pi}\left(\frac{1}{4\nu(t-\tau)}\exp\left(-\beta^2\right) + \frac{1}{2r^2}\left(\exp\left(-\beta^2\right)-1\right)\right)e_k \\ &- \frac{H(t-\tau)}{\pi}\frac{(x_k-w_{0k})r}{r^2}\left(\frac{1}{4\nu(t-\tau)}\exp\left(-\beta^2\right) + \frac{1}{r^2}\left(\exp\left(-\beta^2\right)-1\right)\right),\end{aligned} \tag{2.3}$$

where $r = x-y-w_0(t-\tau)$, $\beta^2 = r^2/4\nu(t-\tau)$, and

$$H(t) = \begin{cases} 1, & t\ge 0, \\ 0, & t<0.\end{cases} \tag{2.4}$$

Here $\delta(x-y)\delta(t-\tau)e_k$ represents the concentrated forces directed along the coordinate axis e_k, $\delta(x-y)$ and $\delta(t-\tau)$ are the Dirac delta functions for the space and time variables, respectively.

Using the usual method [3–5] and the fact that

$$U_k(x-y,0) = 0, \qquad u(x,0) = 0, \qquad \lim_{x\to\infty} U_k(x-y,t-\tau) = 0, \qquad \lim_{x\to\infty} u(x,t) = 0,$$

we obtain

$$\begin{aligned}u_k(x,t) = &-\int_0^t\int_{\Gamma_2} u(y,\tau)\cdot\sigma(U_k,P_k)(x-y,t-\tau)\cdot n(y)\,ds_y\,d\tau \\ &-\int_0^t\int_{\Gamma_2} U_k(x-y,t-\tau)(w_0\cdot n(y))\,ds_y d_\tau + \int_{\Omega_2} U_k(x-y,t)\cdot u_0(y)\,dy \\ &+\int_0^t\int_{\Gamma_2} U_k(x-y,t-\tau)\cdot\lambda(u,p)(y,\tau)\,ds_y\,d\tau, \qquad x\in\Omega_2,\end{aligned} \tag{2.5}$$

$$\begin{aligned}p(x,t) = &-2\nu\int_0^t\int_{\Gamma_2} u(y,\tau)\cdot\nabla Q(x-y,t-\tau)\cdot n(y)\,ds_y\,d\tau \\ &-\int_0^t\int_{\Gamma_2} Q(x-y,t-\tau)(w_0\cdot n(y))\,ds_y\,d\tau + \int_{\Omega_2} Q(x-y,t)\cdot u_0(y)\,dy\tau \\ &+\int_0^t\int_{\Gamma_2} Q(x-y,t-\tau)\cdot\lambda(u,p)(y,\tau)\,ds_y\,d\tau + C, \qquad x\in\Omega_2.\end{aligned} \tag{2.6}$$

$$\begin{aligned}\frac{1}{2}u_k(x,t) = &-\int_0^t\int_{\Gamma_2} u(y,\tau)\cdot\sigma(U_k,P_k)(x-y,t-\tau)\cdot n(y)\,ds_y\,d\tau \\ &-\int_0^t\int_{\Gamma_2} U_k(x-y,t-\tau)(w_0\cdot n(y))\,ds_y\,d\tau + \int_{\Omega_2} U_k(x-y,t)\cdot u_0(y))\,dy \\ &+\int_0^t\int_{\Gamma_2} U_k(x-y,t-\tau)\cdot\lambda(u,p)(y,\tau)\,ds_y\,d\tau, \qquad x\in\Gamma_2,\end{aligned} \tag{2.7}$$

where $Q=(P_1,P_2), C=\text{constant}$, $\lambda(u,p)=\lambda^+(u,p)=\lambda^-(u,p)$, $0\le t\le T$, $k=1,2$.

Therefore, if u and $\lambda(u,p)$ are known on Γ_2, then we can obtain the solution (u,p) on Ω_2 by (2.5),(2.6). Here u and $\lambda(u,p)$ on Γ_2 satisfy the boundary integral equation (2.7). Combining (2.7) and

$$\begin{aligned}\frac{\partial u}{\partial t}-\nabla\cdot\sigma(u,p)+(\psi\cdot\nabla)u+(u\cdot\nabla)\psi+(w_0\nabla)u+(u\cdot\nabla)u=f, &\qquad \text{in } \Omega_1\times(0,T],\\ \operatorname{div} u = 0, &\qquad \text{in } \Omega_1\times(0,T],\\ u=0, &\qquad \text{on } \Gamma\times[0,T],\\ \lambda^-=\lambda^+, &\qquad \text{on } \Gamma_2\times[0,T],\\ u(x,0)=u_0(x), &\qquad \text{in } \Omega_1,\end{aligned} \tag{2.8}$$

we can obtain the coupled variational formulation of problem (S′) in Ω_1.

Let

$$\begin{gathered} H=\left\{v\in L^2(\Omega_1)^2;\ \operatorname{div} v=0,\ v\cdot n|_\Gamma=0\right\},\\ W=\left\{v\in H^1(\Omega_1)^2;\ v|_\Gamma=0\right\},\qquad W_0=\{v\in W;\ \operatorname{div} v=0\},\\ M=\left\{q\in L^2(\Omega_1);\ \int_{\Omega_1} q\,dx=0\right\}.\end{gathered}$$

These spaces, equipped with their usual scalar products and norms, are Hilbert spaces [6]. Moreover, if we set

$$\begin{aligned} a_0(u,v)&=2\nu\sum_{i,j=1}^{2}\int_{\Omega_1}\mathcal{E}_{ij}(u)\mathcal{E}_{ij}(v)\,dx,\\ a_1(u;v,w)&=\int_{\Omega_1}(u\cdot\nabla)v\cdot w\,dx,\\ (p,\operatorname{div} v)&=\int_{\Omega_1}p\operatorname{div} v\,dx,\ (f,v)=\int_{\Omega_1} f\cdot v\,dx,\ \langle v,\lambda\rangle=\int_{\Gamma_2} v\cdot\lambda\,ds_x,\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H^{-1/2}(\Gamma_2)^2$ and $H^{1/2}(\Gamma_2)^2$, it follows from the Green formula that

$$\begin{gathered}\left(\frac{\partial u}{\partial t},v\right)+a_0(u,v)+a_1(u;u,v)+a_1(w_0;u,v)+a(\psi;u,v)\\ +a_1(u;\psi,v)-(p,\operatorname{div} v)+\langle\gamma_0 v,\lambda\rangle=(f,v),\qquad \forall\, v\in W,\\ (q,\operatorname{div} u)=0,\qquad \forall\, q\in L^2(\Omega_1),\\ u(x,0)=u_0(x),\end{gathered} \tag{2.9}$$

or

$$\begin{gathered}\left(\frac{\partial u}{\partial t},v\right)+a_0(u,v)+a_1(u;u,v)+a_1(w_0;u,v)+a_1(\psi;u,v)\\ +a_1(u;\psi,v)+\langle\gamma_0 v,\lambda\rangle=(f,v),\qquad \forall\, v\in W_0,\\ u(x,0)=u_0(x),\end{gathered} \tag{2.10}$$

where $\gamma_0 v = v|_{\Gamma_2}$. On the other hand, we define the Hilbert spaces

$$H_0^{1/2}(\Gamma_2)^2 = \left\{w \in H^{1/2}(\Gamma_2)^2;\ \langle n, w\rangle = 0\right\},$$
$$\Lambda = H_0^{-1/2}(\Gamma_2)^2 = \left\{\mu \in H^{-1/2}(\Gamma_2)^2;\ \langle n, \mu\rangle = 0\right\}.$$

Clearly, it holds that $\lambda = \sigma(u,p)\cdot n|_{\Gamma_2} \in \Lambda$.

Equation (2.7) formally multiplied by $\mu \in \Lambda$ and integrated over Γ_2 yields

$$b(t,\lambda,\mu) - \frac{1}{2}\langle \gamma_0 u, \mu\rangle + \langle G(t,\gamma_0 u), \mu\rangle + \langle K(t,u_0),\mu\rangle = 0, \qquad \forall\, \mu \in \Lambda, \tag{2.11}$$

where $U = (U_1, U_2)$, $k = 1,2$,

$$\begin{aligned}
b(t,\lambda,\mu) &= \int_0^t \int_{\Gamma_2}\int_{\Gamma_2} \mu(x)\cdot U(x-y,t-\tau)\cdot\lambda(y,\tau)\,ds_y\,ds_x\,d\tau,\\
K_i(t,u_0) &= \int_{\Omega_2} U_i(x-y,t)\cdot u_0(y)\,dy,\\
G_k(t,\gamma_0 u) &= -\int_0^t\int_{\Gamma_2} u(y,\tau)\cdot\sigma(U_k,P_k)(x-y,t-\tau)\cdot n(y)\,ds_y\,d\tau\\
&\quad -\int_0^t\int_{\Gamma_2} U_i(x-y,t-\tau)w_0\cdot n(y)\,ds_y\,d\tau.
\end{aligned}$$

Finally, by combining (2.9) or (2.10) and (2.11), we obtain the coupled variational formulation of problem (S′):

$$\begin{aligned}
&\text{find } (u(t),p(t),\lambda(t)) \in W\times M\times\Lambda,\ 0\le t\le T, \quad \text{such that}\\
&\left(\frac{\partial u}{\partial t}, v\right) + a_0(u,v) + a_1(u;u,v) + \langle\gamma_0 v,\lambda\rangle + a_1(w_0;u,v)\\
&+a_1(u;\psi,v) + a_1(\psi;u,v) - (p,\operatorname{div} v) = (f,v), \qquad \forall\, v\in W,\\
&b(t,\lambda,\mu) - \frac{1}{2}\langle\gamma_0 u,\mu\rangle + \langle G(t,\gamma_0 u),\mu\rangle + \langle K(t,u_0),\mu\rangle = 0, \qquad \forall\,\mu\in\Lambda,\\
&(q,\operatorname{div} u) = 0, \qquad \forall\, q\in L^2(\Omega_1),\\
&u(0) = u_0,
\end{aligned} \tag{Q}$$

or

$$\begin{aligned}
&\text{find } (u(t),\lambda(t)) \in W_0\times\Lambda,\ 0\le t\le T, \quad \text{such that}\\
&\left(\frac{\partial u}{\partial t}, v\right) + a_0(u,v) + a_1(u;u,v) + \langle\gamma_0 v,\lambda\rangle\\
&+a_1(w_0;u,v) + a_1(\psi;u,v) + a_1(u;\psi,v) = (f,v), \qquad \forall\, v\in W_0,\\
&b(t,\lambda,\mu) - \frac{1}{2}\langle\gamma_0 u,\mu\rangle + \langle G(t,\gamma_0 u),\mu\rangle + \langle K(t,u_0),\mu\rangle = 0, \qquad \forall\,\mu\in\Lambda,\\
&u(0) = u_0.
\end{aligned} \tag{P}$$

Referring again to [1–3,5,7–9], we can prove that the following estimates hold:

$$|a_0(u,v)| \le 3\nu|u|_1|v|_1, \quad a(u,u)\ge\alpha|u|_1^2, \qquad \forall\, u,v\in W, \tag{2.12}$$
$$|a_1(u;v,w)| \le c_0(|u|_0|u|_1)^{1/2}|v|_1(|w|_0|w|_1)^{1/2}, \qquad \forall\, u,v,w\in W, \tag{2.13}$$
$$\int_0^T b(t,\lambda,\mu)\,dt \le c_0\|\lambda\|_{L^2(0,T;\Lambda)}\|\mu\|_{L^2(0,T;\Lambda)}, \qquad \forall\,\lambda,\mu\in L^2(0,T;\Lambda), \tag{2.14}$$
$$\int_0^T b(t,\lambda,\lambda)\,dt \ge c_1\|\lambda\|^2_{L^2(0,T;\Lambda)}, \qquad \forall\,\lambda\in L^2(0,T;\Lambda), \tag{2.15}$$
$$|a_1(w_0;u,v)| \le |w_0|\,|u|_1|v|_0, \qquad \forall\, u,v\in W, \tag{2.16}$$
$$|a_1(\psi;u,v) + a_1(u;\psi,v)| \le \frac{\alpha}{16}|u|_1|v|_1, \qquad \forall\, u,v\in W, \tag{2.17}$$

where $\alpha > 0$ and c_i $(i = 0, 1, \dots)$ are positive constants dependent on Ω_1, and

$$|u|_0 = \|u\|_{L^2(\Omega_1)^2}, \qquad |u|_1 = \|\nabla u\|_{L^2(\Omega_1)^4}.$$

REMARK. The proof of (2.17) is similar to the ones in [1], so it can be omitted. The following important result is similar to the ones in [3].

LEMMA 2.1. *For any $u \in L^2(0,T;W)$ with $\int_{\Gamma_2} u \cdot n\, ds_x = 0$, we have the following variational formulation. Find $\lambda \in L^2(0,T;\Lambda)$ such that*

$$\begin{aligned}\int_0^T b(t,\lambda,\mu)\,dt - \frac{1}{2}\int_0^T \langle \gamma_0 u, \mu\rangle\, dt + \int_0^T \langle G(t,\gamma_0 u),\mu\rangle\, dt \\ + \int_0^T \langle K(t,u_0),\mu\rangle\, dt = 0, \qquad \forall\, \mu \in L^2(0,T,\Lambda),\end{aligned} \tag{2.18}$$

admits a unique solution $\lambda = \lambda(\gamma_0 u, u_0)$ satisfying

$$\|\lambda(\gamma_0 u, u_0)\|_{L^2(0,T;\Lambda)} \le c_3 \left(\|u\|_{L^2(0,T;W)} + |u_0|_{0,\Omega_2}\right), \tag{2.19}$$

$$\int_0^t \left[\langle \gamma_0 u, \lambda(\gamma_0 u, u_0)\rangle + a_1(w_0;u,u)\right] d\tau \ge -\frac{1}{2}|u_0|^2_{0,\Omega_2}, \qquad \forall\, 0 \le t \le T. \tag{2.20}$$ ∎

3. WELL POSEDNESS AND REGULARITY

In this section, we aim to provide the well posedness and regularity results of problem (Q). First, we recall the following lemma in [9].

LEMMA 3.1. *Under assumption (1.1), if $u_0 \in H$ and $f \in L^2(0,T;H^{-1}(\Omega_1)^2)$, then the weak formulation of problem (S′),*

$$\begin{gathered}\text{find } (u(t),p(t)) \in H_0^1(\Omega)^2 \times \frac{L^2_{\text{loc}}(\Omega)}{R}, \qquad 0 \le t \le T, \text{ such that} \\ \left(\frac{\partial u}{\partial t}, v\right)_\Omega + \nu(\nabla u, \nabla v)_\Omega + a_1(u;u,v) + ((w_0\cdot\nabla)u,v)_\Omega \\ + a_1(\psi;u,v) + a_1(u;\psi,v) - (p,\operatorname{div} v)_\Omega = (f,v), \qquad \forall\, v \in H_0^1(\Omega)^2, \\ u(0) = u_0\end{gathered} \tag{3.1}$$

admits a unique solution $(u,p) \in L^\infty(0,T;H(\Omega)) \cap L^2(0,T;H_0^1(\Omega)^2) \times D'(0,T;L^2_{\text{loc}}(\Omega)/R)$ satisfying

$$|u(t)|^2_{0,\Omega} + \alpha\int_0^t |u|^2_{1,\Omega}\, d\tau \le \exp\left(\frac{2}{\alpha}\varepsilon^2 t\right)\left\{|u_0|^2_{0,\Omega} + \frac{2}{\alpha}\int_0^t |f|^2_{-1}\, d\tau\right\}, \tag{3.2}$$

where

$$\begin{gathered}H(\Omega) = \left\{v \in L^2(\Omega)^2;\ \operatorname{div} v = 0 \text{ in } \Omega \text{ and } v\cdot n|_\Gamma = 0\right\}, \\ |f|_{-1} = \sup_{v\in W}\frac{(f,v)}{|v|_1}, \qquad (u,v)_\Omega = \int_\Omega u\cdot v\, dx.\end{gathered}$$ ∎

Now, we give the well posedness of problem (Q).

THEOREM 3.2. *Under the assumptions of Lemma 3.1, the variational formulation (Q) admits a unique solution $(u,p,\lambda) \in L^\infty(0,T;H) \cap L^2(0,T;W_0) \times D'(0,T;M) \times L^2(0,T;\wedge)$ and $\lambda = \lambda(\gamma_0 u, u_0)$ satisfies (2.19), (2.20).*

PROOF. According to Lemma 3.1, problem (S′) admits a unique solution $(u,p) \in L^\infty(0,T;H(\Omega)) \cap L^2(0,T;H_0^1(\Omega)^2) \times D'(0,T;L^2_{\text{loc}}(\Omega)/R)$. Recalling the discussions of Section 2, we know that

$(u|_\Omega, p|_\Omega, \sigma(u,p)\cdot n|_{\Gamma_2}) \in L^\infty(0,T;H)\cap L^2(0,T;W_0)\times D'(0,T;M)\times L^2(0,T;\Lambda)$ satisfies problem (Q).

Next, we shall prove that the solution to (Q) is unique. In fact, if (u_1,p_1,λ_1) and (u_2,p_2,λ_2) are two solutions of (Q), then $(w,\eta,\mu)=(u_1-u_2,p_1-p_2,\lambda_1-\lambda_2)$ satisfies

$$\begin{aligned}\left(\frac{\partial w}{\partial t},v\right)&+a_0(w,v)+a_1(w;u_1,v)+a_1(u_2;w,v)+a_1(w_0;w,v)+a_1(\psi;w,v)\\&+a_1(w;\psi,v)-(\eta,\operatorname{div}v)+\langle\gamma_0 v,\mu\rangle=0,\qquad\forall v\in W,\end{aligned}\tag{3.3}$$

$$(q,\operatorname{div}w)=0,\qquad\forall q\in L^2(\Omega_1),\tag{3.4}$$

$$b(t,\mu,\mu')-\frac{1}{2}\langle\gamma_0 w,\mu'\rangle+\langle G(t,\gamma_0 w),\mu'\rangle=0,\qquad\forall\mu'\in\Lambda,\tag{3.5}$$

$$w(0)=0.\tag{3.6}$$

According to Lemma 2.1, $\mu=\lambda(\gamma_0 w,0)$ satisfies

$$\int_0^t[\langle\gamma_0 w,\lambda(\gamma_0 w,0)\rangle+a_1(w_0,w,w)]\,d\tau\ge 0,\qquad\forall\,0\le t\le T,\tag{3.7}$$

$$\|\lambda(\gamma_0 w,0)\|_{L^2(0,T;\Lambda)}\le c_3\|w\|_{L^2(0,T;W)}.\tag{3.8}$$

Taking $v=w$ in (3.3) and using (3.4), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|w|_0^2&+a_0(w,w)+a_1(w;u_1,w)+a_1(u_2;w,w)+a_1(\psi;w,w)\\&+a_1(w;\psi,w)+a_1(w_0;w,w)+\langle\gamma_0 w,\lambda(\gamma_0 w,0)\rangle=0.\end{aligned}\tag{3.9}$$

Thanks to (2.12), (2.13), and (2.17), we have

$$a_0(w,w)\ge\alpha|w|_1^2,\qquad|a_1(w;u_1,w)|\le\frac{\alpha}{4}|w|_1^2+\alpha^{-1}c_0^2|u_1|_1^2|w|_0^2,\tag{3.10}$$

$$|a_1(u_2;w,w)|\le c_0|u_2|_0^{1/2}|u_2|_1^{1/2}|w|_1^{3/2}|w|_0^{1/2}\le\frac{\alpha}{4}|w|_1^2+\left(\frac{2}{\alpha}\right)^3c_0^4|u_2|_0^2|u_2|_1^2|w|_0^2,\tag{3.11}$$

$$|a_1(\psi;w,w)|+a_1(w,\psi,w)|\le\frac{\alpha}{4}|w|_1^2.\tag{3.12}$$

Combining (3.9) with (3.10)–(3.12) yields

$$\frac{d}{dt}|w|_0^2+\frac{\alpha}{2}|w|_1^2+2a_1(w_0;w,w)+2\langle\gamma_0 w,\lambda(\gamma_0 w,0)\rangle\le g(t)|w|_0^2,\tag{3.13}$$

where

$$g(t)=\frac{2}{\alpha}c_0^2|u_1|_1^2+2\left(\frac{2}{\alpha}\right)^3c_0^4|u_2|_0^2|u_2|_1^2\in L^1(0,T).$$

Integrating (3.13) and using (3.7), we obtain

$$|w(t)|_0^2+\frac{\alpha}{2}\int_0^t|w|_1^2\,d\tau\le|w(0)|_0^2+\int_0^t g(\tau)|w(\tau)|_0^2\,d\tau.\tag{3.14}$$

Applying the Gronwall lemma and the fact that $w(0)=0$, we obtain

$$|w(t)|_0^2+\frac{\alpha}{2}\int_0^t|w|_1^2\,d\tau\le 0.\tag{3.15}$$

Hence, one finds

$$\|w\|_{L^2(0,T;W)}=0.\tag{3.16}$$

Combining (3.16) and (3.8) yields

$$\|\lambda(\gamma_0 w, 0)\|_{L^2(0,T;\Lambda)} = 0. \tag{3.17}$$

Recalling the inf-sup condition (see [10]): there exists a constant $\beta_1 = \beta_1(\Omega_1) > 0$ dependent on Ω_1 such that

$$\beta_1 |q|_0 \leq \sup_{v \in H_0^1(\Omega_1)^2} \frac{(q, \operatorname{div} v)}{|v|_1}, \qquad \forall q \in M, \tag{3.18}$$

we derive, from (3.3), (3.16), and (3.18), that

$$|\eta(t)|_0^2 \leq 0, \qquad 0 \leq t \leq T. \tag{3.19}$$

Thus, (3.16), (3.17), and (3.19) have completed the proof of the uniqueness. ∎

THEOREM 3.3. *Under assumption (1.1), if $u_0 \in W_0, f \in L^2(0,T;L^2(\Omega_1)^2)$, then the solution (u,p,λ) of (Q) satisfies the following regularity:*

$$\begin{aligned}&\|u\|^2_{L^\infty(0,T;W)} + \|u\|^2_{L^2(0,T;H^2(\Omega)^2)} + \|\lambda\|^2_{L^2(0,T;H_0^{1/2}(\Gamma_2)^2)} \\ &+ \left\|\frac{\partial u}{\partial t}\right\|_{L^2(0,T;H)} + \|p\|^2_{L^2(0,T;H^1(\Omega_1))} \leq c\left(|u_0|_1^2 + \|f\|^2_{L^2(0,T;L^2(\Omega_1)^2)}\right),\end{aligned} \tag{3.20}$$

where $c = c(\alpha, \Omega_1, T)$ is a positive constant dependent on α, Ω_1 and T, $\|\cdot\|_m$ denote the norm on $H^m(\Omega_1)$(or $H^m(\Omega_1)^2$), $|\cdot|_{m,\Omega}$ denotes the norm on $H_0^m(\Omega)^2$. In particular, the norm $\|\cdot\|_m$ is equivalent to the seminorm $|\cdot|_m$ on the Sobolev space $H^m(\Omega_1)^2 \cap W$ (see [11]).

PROOF. Let us introduce the Stokes operator A in $H(\Omega)$. We denote by P the orthogonal projection operator from $L^2(\Omega)^2$ onto $H(\Omega)$. Then the Stokes operator A is defined by $A = -P\Delta$ with domain $D(A) = H^2(\Omega)^2 \cap W_0(\Omega)$, where $W_0(\Omega) = \{v \in H_0^1(\Omega)^2; \operatorname{div} v = 0 \text{ in } \Omega\}$. Moreover, it follows from [2,7] that

$$\|u\|^2_{2,\Omega} \leq c_4\left(|u|^2_{0,\Omega} + |Au|^2_{0,\Omega}\right), \qquad \forall u \in D(A), \tag{3.21}$$

$$\|u\|_{L^\infty(\Omega_1)^2} \leq c_5 |u|_0^{1/2} \|u\|_2^{1/2}, \qquad \forall u \in D(A), \tag{3.22}$$

$$\|u\|^2_{L^4(\Omega)^2} \leq 2^{1/2} |u|_{0,\Omega} |u|_{1,\Omega}, \qquad \forall u \in H_0^1(\Omega)^2. \tag{3.23}$$

Applying the projection operator P to both sides of problem (S′), we get formally

$$\frac{\partial u}{\partial t} + \nu A u + X_{\Omega_1} P((u\cdot\nabla)u + (u\cdot\nabla)\psi + (\psi\cdot\nabla)u) + P(w_0\cdot\nabla)u = Pf, \tag{3.24}$$

$$u(0) = u_0. \tag{3.25}$$

Now, let us take the scalar product in $H(\Omega)$ of equation (3.24) with Au, then

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|u|^2_{1,\Omega} + \nu|Au|^2_{0,\Omega} + a_1(u; u+\psi, Au) + a_1(\psi; u, Au) \\ &+((w_0\cdot\nabla)u, Au)_\Omega = (f, Au) \leq \frac{\nu}{4}|Au|^2_{0,\Omega} + \nu^{-1}|f|_0^2,\end{aligned} \tag{3.26}$$

where $(u,v)_\Omega = \int_\Omega u\cdot v\,dx$. Thanks to (3.21)–(3.23) and the Young inequality, we have

$$\begin{aligned}|((w_0\cdot\nabla)u, Au)_\Omega| &\leq |w_0|\,|u|_{1,\Omega}|Au|_{0,\Omega} \leq \frac{\nu}{16}|Au|^2_{0,\Omega} + \frac{4}{\nu}|w_0|^2|u|^2_{1,\Omega}, \\ |a_1(u; u+\psi, Au)| &\leq \|u\|_{L^\infty(\Omega_1)^2}|u+\psi|_1|Au|_{0,\Omega} \\ &\leq \frac{\nu}{8}|Au|^2_{0,\Omega} + \frac{\nu}{16}|u|^2_{0,\Omega} + \frac{c_6}{4}|u|^2_{0,\Omega}\left(|u|^4_{1,\Omega} + |\psi|_1^4\right), \\ |a_1(\psi; u, Au)| &\leq \|\psi\|_{L^\infty(\Omega_1)^2}|u|_{1,\Omega}|Au|_{0,\Omega} \\ &\leq \frac{\nu}{16}|Au|^2_{0,\Omega} + \frac{c_6}{4}\|\psi\|_2^2|u|^2_{1,\Omega}.\end{aligned} \tag{3.27}$$

Combining (3.26) and (3.27) yields

$$\frac{d}{dt}|u|_{1,\Omega}^2 + \nu|Au|_{0,\Omega}^2 \le F(t) + g(t)|u(t)|_{1,\Omega}^2, \tag{3.28}$$

where

$$F(t) = \frac{2}{\nu}|f(t)|_0^2 + \frac{\nu}{2}|u(t)|_{0,\Omega}^2 + c_0|\psi|_1^2|u(t)|_{0,\Omega}^2,$$
$$g(t) = \frac{8}{\alpha}|w_0|^2 + c_0|u(t)|_{1,\Omega}^2 + c_0\|\psi\|_2^2.$$

Integrating (3.28), we obtain

$$|u(t)|_{1,\Omega}^2 + \alpha\int_0^t |Au|_{0,\Omega}^2\,d\tau \le |u_0|_{1,\Omega}^2 + \int_0^t F(\tau)\,d\tau + \int_0^t g(\tau)|u(\tau)|_{1,\Omega}^2\,d\tau. \tag{3.29}$$

Applying the Gronwall lemma, we obtain

$$|u(t)|_{1,\Omega}^2 + \alpha\int_0^t |Au|_{0,\Omega}^2\,d\tau \le \exp\left(\int_0^T g(\tau)\,d\tau\right)\left\{|u_0|_{1,\Omega}^2 + \int_0^T F(\tau)\,d\tau\right\}. \tag{3.30}$$

According to (3.2), we know that $\exp(\int_0^t g(\tau)\,d\tau)$ and $\int_0^T F(\tau)\,d\tau$ are bounded. Hence, we imply

$$u \in L^\infty(0,T;W_0(\Omega)) \cap L^2\left(0,T;H^2(\Omega)^2 \cap W_0(\Omega)\right). \tag{3.31}$$

Using (3.24) again, we have

$$\begin{aligned}\int_0^T \left|\frac{\partial u}{\partial t}\right|_{0,\Omega}^2 dt &\le c\int_0^T \left(|Au|_{0,\Omega}^2 + |f|_0^2 + \|u\|_{1,\Omega}^2 + \|u\|_{L^\infty(\Omega)^2}^2|u|_{1,\Omega}^2\right) dt \\ &\le c\int_0^T \left(|Au|_{0,\Omega}^2 + |f|_0^2 + |u|_{0,\Omega}^2 + \|u\|_{1,\Omega}^2 + |u|_{0,\Omega}|u|_{1,\Omega}^4\right) dt.\end{aligned} \tag{3.32}$$

Hence, (3.30) and (3.32) imply

$$\|u(t)\|_{L^\infty(0,T;W)}^2 + \|u\|_{L^2(0,T;H^2(\Omega_1)^2)}^2 + \left\|\frac{\partial u}{\partial t}\right\|_{L^2(0,T;H)}^2 \le c\left(|u_0|_{1,\Omega}^2 + \int_0^T |f|_0^2\,d\tau\right). \tag{3.33}$$

Using again the trace theorem (see [12]), we have

$$\begin{aligned}\|\lambda\|_{L^2\left(0,T;H_0^{1/2}(\Gamma_2)^2\right)} &= \|\sigma(u,p)\cdot n|_{\Gamma_2}\|_{L^2\left(0,T;H_0^{1/2}(\Gamma_2)^2\right)}^2 \\ &\le c\left\{\|u\|_{L^2(0,T;H^2(\Omega_1)^2)}^2 + \|p\|_{L^2(0,T;H^1(\Omega_1))}^2\right\}.\end{aligned} \tag{3.34}$$

Finally, from problem (Q) and (3.18), we deduce

$$\begin{aligned}|p(t)|_0 &\le \beta_1^{-1}\sup_{v\in H_0^1(\Omega_1)^2}\frac{(p(t),\operatorname{div} v)}{|v|_1} \\ &\le \beta_1^{-1}\left(\gamma\left|\frac{\partial u}{\partial t}\right|_0 + 3\alpha|u|_1 + \gamma|u|_1^2 + \gamma|w_0|\,|u|_1 + 2\gamma|\psi|_1|u|_1 + |f|_{-1}\right),\end{aligned} \tag{3.35}$$

where $\gamma = \gamma(\Omega_1)$ is the Poincaré constant dependent on Ω_1 such that

$$|u|_0 \le \gamma|u|_1, \qquad \forall\, u \in W. \tag{3.36}$$

Moreover, according to problem (S′), we have

$$|\nabla p(t)|_0 \le \left|\frac{\partial u}{\partial t}\right|_0 + \alpha\|u\|_2 + \|u\|_{L^\infty(\Omega_1)}(|u|_1 + |\psi|_1) + \|\psi\|_{L^\infty(\Omega_1)^2}|u|_1 + |f|_0. \tag{3.37}$$

Recalling (3.33), we derive from (3.34)–(3.37) that

$$\int_0^T \|p\|_1^2\,dt + \int_0^T \|\lambda\|_{H^{1/2}(\Gamma_2)^2}^2\,dt \le c\left(|u_0|_{1,\Omega}^2 + \int_0^T |f|_0^2\,dt\right). \tag{3.38}$$

Thus, (3.33) and (3.38) imply (3.20), the proof ends. ∎

4. THE APPROXIMATE COUPLED PROBLEM

For simplicity, here we restrict the discussion to the case where Ω_1 has polygonal boundaries, but the results can be extended to the general curvilinear domain, by introducing an approximate boundary $\Gamma_h \cup \Gamma_{2h}$. For further details, we also refer to [13].

The first step consists of introducing the finite-dimensional subspaces such that

$$X_h \subset H^1(\Omega_1), \qquad S_h \subset H^{-1/2}(\Gamma_2), \qquad \bar{M}_h \subset L^2(\Omega_1).$$

We define

$$W_h = X_h^2 \cap W, \qquad \Lambda_h = S_h^2 \cap \Lambda, \qquad X_{0h} = X_h \cap H_0^1(\Omega_1), \qquad M_h = \bar{M}_h \cap M.$$

In addition, we introduce

$$W_{0h} = \left\{ v_h \in W_h; (q_h, \operatorname{div} v_h) = 0,\ \forall\, q_h \in \bar{M}_h \right\}.$$

With these spaces, problem (Q) can be approximated by the following problem. Find $(u_h(t), p_h(t), \lambda_h(t)) \in W_h \times M_h \times \Lambda_h$, $0 \le t \le T$, such that

$$\begin{aligned}
&\left(\frac{\partial u_h}{\partial t}, v\right) + a_0(u_h, v) + a_1(u_h; u_h, v) + \langle \gamma_0 v, \lambda_h\rangle \\
&+a_1(w_0; u_h, v) + a_1(\psi; u_h, v) + a_1(u_h; \psi, v) - (p_h, \operatorname{div} v) = (f, v), \qquad \forall\, v \in W_h, \\
&b(t, \lambda_h, \mu) - \frac{1}{2}\langle \gamma_0 u_h, \mu\rangle + \langle G(t, \gamma_0 u_h), \mu\rangle + \langle K(t, u_0), \mu\rangle = 0, \qquad \forall\, \mu \in \Lambda_h, \\
&(q, \operatorname{div} u_h) = 0, \qquad \forall\, q \in \bar{M}_h, \\
&u_h(0) = P_h u_0,
\end{aligned} \tag{Q_h}$$

where $P_h : L^2(\Omega_1)^2 \to W_h$ is the L^2-orthogonal projection operator defined by

$$(P_h u, v_h) = (u, v_h), \qquad \forall\, v_h \in W_h, \quad u \in L^2(\Omega_1)^2.$$

Moreover, $\rho_h : L^2(\Omega_1)^2 \to \bar{M}_h$ is also the L^2-orthogonal projection operator defined by

$$(\rho_h p, q_h) = (p, q_h), \qquad \forall\, q_h \in \bar{M}_h, \quad p \in L^2(\Omega_1).$$

Obviously, P_h is also the L^2-orthogonal operator from H onto W_{0h}. We require the following approximation assumptions.

(H_1) There exists an operator $I_h : H^2(\Omega_1)^2 \cap W \to W_h$ such that

$$\begin{aligned}
(q_h, \operatorname{div}(v - I_h v)) &= 0, \qquad \forall\, q_h \in \bar{M}_h, \quad v \in H^2(\Omega_1)^2, \\
|v - I_h v|_0 &\le ch\|v\|_{2,\Omega_1}, \qquad \forall\, v \in H^2(\Omega_1)^2.
\end{aligned}$$

(H_2) The orthogonal projection operator $s_h : L^2(\Gamma_2)^2 \to \Lambda_h$ satisfies

$$\|\mu - s_h\mu\|_\Lambda \le ch\|\mu\|_{H^{1/2}(\Gamma_2)^2}, \qquad \forall\, \mu \in H^{1/2}(\Gamma_2)^2.$$

(H_3) The L^2-orthogonal operator ρ_h satisfies

$$|q - \rho_h q|_0 \le ch\|q\|_1, \qquad \forall\, q \in H^1(\Omega_1)^2 \cap M.$$

(H_4) There exists a constant $\beta_* > 0$, independent of h such that

$$\beta_* |q_h|_0 \le \sup_{v_h \in X_{0h}^2} \frac{(q_h, \operatorname{div} v_h)}{|v_h|_1}, \qquad \forall\, q_h \in M_h.$$

We now give some examples of subspaces W_h, Λ_h, and M_h such that Assumptions (H_1)–(H_4) are satisfied. Let $\{\tau_h\}, h > 0$, be a uniformly regular family of Ω_1 made of triangles K with diameters bounded by h. For any integer m, we denote by P_m the space of polynomials of degree less than or equal to m. Let us denote by s_i, $i = 1, \dots, N, (|s_i| \le h)$, the finite number of segments of a line composing the boundary Γ_2.

EXAMPLE. (See [5,10].)

$$
\begin{aligned}
X_h &= \left\{v_h \in C^0\left(\bar{\Omega}_1\right);\ v_h|_K \in P_2,\ \forall K \in \tau_h\right\}, \\
S_h &= \left\{w_h \in L^2(\Gamma_2);\ w_h|_{s_i} \in P_0,\ 1 \le i \le N\right\}, \\
\bar{M}_h &= \left\{q_h \in L^2(\Omega_1);\ q_h|_K \in P_0,\ \forall K \in \tau_h\right\}.
\end{aligned}
$$

For the finite element spaces W_h and $\bar{M}_h$ constructed above, the following fact holds:

$$\int_{\Gamma_2} v_h \cdot n \, ds_x = 0, \qquad \forall v_h \in W_{0h}. \tag{4.1}$$

In fact, for $v_h \in W_{0h}$, there holds

$$\int_{\Omega_1} q_h \operatorname{div} v_h \, dx = 0, \qquad \forall q_h \in \bar{M}_h. \tag{4.2}$$

In particular, for each $K \in \tau_h$, we take

$$q_h = \begin{cases} 1, & x \in K, \\ 0, & x \bar{\in} K. \end{cases}$$

Thus, (4.2) yields

$$\int_K \operatorname{div} v_h \, dx = \int_{\partial K} v_h \cdot n \, ds_x = 0. \tag{4.3}$$

Summing (4.3) for $K \in \tau_h$, one finds

$$\int_{\Gamma_2} v_h \cdot n \, ds_x = \int_{\Gamma \cup \Gamma_2} v_h \cdot n \, ds_x = \sum_{K \in \tau_h} \int_{\partial K} v_h \cdot n \, ds_x = 0. \tag{4.4}$$

The proof ends. ∎

The following properties which are classical consequences of (H_1)–(H_4) (see [8,10]) will be useful:

$$
\begin{aligned}
|P_h v|_1 &\le c|v|_1, && \forall v \in W, && (4.5)\\
|v - P_h v| + h|v - P_h v|_1 &\le ch^2 \|v\|_2, && \forall v \in H^2(\Omega_1)^2 \cap W, && (4.6)\\
|v - P_h v|_0 &\le ch|v|_1, && \forall v \in W, && (4.7)\\
|q - \rho_h q| &\le ch\|q\|_1, && \forall q \in H^1(\Omega_1) \cap M. && (4.8)
\end{aligned}
$$

Thanks to (4.1), it is easy to prove the following result.

LEMMA 4.1. *For any $u_h \in L^2(0,T;W_{0h})$, the variational formulation: find $\lambda_h \in L^2(0,T;\Lambda_h)$ such that*

$$
\begin{aligned}
&\int_0^T b(t, \lambda_h, \mu)\, dt - \frac{1}{2}\int_0^T \langle \gamma_0 u_h, \mu\rangle \, dt + \int_0^T \langle K(t, u_0), \mu\rangle \, dt \\
&\qquad + \int_0^T \langle G(t, \gamma_0 u_h), \mu\rangle \, dt = 0, \qquad \forall \mu \in L^2(0,T;\Lambda_h),
\end{aligned} \tag{4.9}
$$

admits a unique solution $\lambda_h = \lambda_h(\gamma_0 u_h, u_0) \in L^2(0,T;\Lambda_h)$ which satisfies

$$\|\lambda(\gamma_0 u_h, u_0)\|_{L^2(0,T;\Lambda)} \le c_7 \left(\|u_h\|_{L^2(0,T;W)} + |u_0|_{0,\Omega}\right), \tag{4.10}$$

$$\int_0^t \left[a_1(w_0; u_h, u_h) + \langle \gamma_0 u_h, \lambda_h(\gamma_0 u_h, u_0)\rangle\right] d\tau \ge -\frac{1}{2}|u_0|^2_{0,\Omega_2}, \qquad \forall\, 0 \le t \le T. \tag{4.11}$$ ∎

This proof is similar to the ones of Lemma 2.1, so it can be omitted.

Due to Lemma 4.1 and Assumptions (H_1)–(H_4), the existence and uniqueness of the approximate solution (u_h, p_h, λ_h) can be shown to those of the exact solution (u, p, λ) of problem (Q).

5. ERROR ESTIMATES

In this section, we aim to derive error estimates for the coupling methods of boundary integral and finite element presented in Section 4.

THEOREM 5.1. *Under assumption (1.1), if $u_0 \in H(\Omega)$ and $f \in L^2(0,T;H^{-1}(\Omega_1)^2)$, then the solution (u,p) of problem (S′) satisfies*

$$|\bar{u}(t)-u(t)|_{0,\Omega}^2+\alpha\int_0^t|\bar{u}-u|_{1,\Omega}^2\,d\tau\le C_1(\alpha,T,f,u_0)\varepsilon^2\int_0^T\delta^2(t)\,dt, \tag{5.1}$$

where

$$\delta(t)=\frac{|u(t)|_{1,\Omega_2}}{|u(t)|_{1,\Omega}}\le 1,$$

and $C_1(\alpha,T,f,u_0)$ is a positive constant depends on α, T, f, and u_0.

PROOF. From problem (S), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\bar{u}|_{0,\Omega}^2+\nu|\bar{u}|_{1,\Omega}^2+\int_\Omega((\bar{u}+w_0)\cdot\nabla)\bar{u}\cdot\bar{u}\,dx+a_1(\bar{u};\psi,\bar{u})+a_1(\psi;\bar{u},u)&=(f,\bar{u})_{\Omega_1}\\ \le\frac{\nu}{4}|\bar{u}|_{1,\Omega}^2+\frac{1}{\nu}|f|_{-1}^2.&\end{aligned} \tag{5.2}$$

Recalling Temam [1] and (2.17), we have

$$\int_\Omega((\bar{u}+w_0)\cdot\nabla)\bar{u}\cdot\bar{u}\,dx=0,\ |a_1(\bar{u};\psi,u)+a_1(\psi;\bar{u},\bar{u})|\le\frac{\nu}{4}|\bar{u}|_{1,\Omega}^2. \tag{5.3}$$

Hence, (5.2) and (5.3) yield

$$\frac{d}{dt}|\bar{u}|_{0,\Omega}^2+\nu|\bar{u}|_{1,\Omega}^2\le 2\nu^{-1}|f|_{-1}^2. \tag{5.4}$$

Integrating (5.4), we obtain

$$|\bar{u}|_{0,\Omega}^2+\nu\int_0^t|\bar{u}|_{1,\Omega}^2\,d\tau\le|u_0|_{0,\Omega}^2+\nu^{-1}\int_0^T|f|_{-1}^2\,d\tau. \tag{5.5}$$

Moreover, we derive from problems (S) and (S′) that

$$\begin{aligned}\frac{d}{dt}|E|_{0,\Omega}^2+\alpha|E|_{1,\Omega}^2+\int_\Omega(E\cdot\nabla)\bar{u}\cdot E\,dx+\int_{\Omega_2}(u\cdot\nabla)u\cdot E\,dx&\\ +a_1(\psi;E,E)+a_1(E;\psi,E)=0,&\end{aligned} \tag{5.6}$$

where $E=\bar{u}-u$ and $((w_0\cdot\nabla)E,E)_\Omega=0$ (see [1]). Thanks to (1.1), (2.17), (3.23), and (5.3), we have

$$\begin{aligned}&|a_1(\psi;E,E)+a_1(E;\psi,E)|\le\frac{\nu}{4}|E|_{1,\Omega}^2,\\ &\left|\int_\Omega(E\cdot\nabla)\bar{u}\cdot E\,dx\right|\le\|E\|_{L^4(\Omega)^2}^2|\bar{u}|_{1,\Omega}\le\frac{\nu}{4}|E|_{1,\Omega}^2+\nu^{-1}|\bar{u}|_{1,\Omega}^2|E|_{0,\Omega}^2,\end{aligned} \tag{5.7}$$

$$\begin{aligned}\left|\int_{\Omega_2}((u\cdot\nabla)u\cdot E\,dx)\right|&\le\varepsilon|u|_{1,\Omega_2}|E|_{0,\Omega_2}\le\varepsilon\delta|u|_{1,\Omega}|E|_{0,\Omega}\\ &\le\nu^{-1}|u|_{1,\Omega}^2|E|_{0,\Omega}^2+\frac{1}{4}\nu\varepsilon^2\delta^2.\end{aligned} \tag{5.8}$$

Combining (5.6) with (5.7),(5.8) yields

$$\frac{d}{dt}|E|_{0,\Omega}^2\le\frac{\nu}{2}\varepsilon^2\delta^2+(2\nu)^{-1}\left(|u|_{1,\Omega}^2+|\bar{u}|_{1,\Omega}^2\right)|E|_{0,\Omega}^2. \tag{5.9}$$

Integrating (5.9), we obtain (5.1). The proof ends. ∎

THEOREM 5.2. *Under assumptions (1.1) and (H$_1$)–(H$_4$), if* $u_0 \in W_0$, $f \in L^2(0,T;L^2(\Omega_1)^2)$, *then the approximate solution* (u_h, p_h, λ_h) *of problem* (Q_h) *satisfies*

$$|u(t) - u_h(t)|_0^2 + \alpha \int_0^t |u - u_h|_1^2 \, d\tau \le C_2(\alpha, \Omega_1, T, f, u_0) h^2, \tag{5.10}$$

$$\int_0^T \|\lambda - \lambda_h\|^2 \, dt \le C_2(\alpha, \Omega_1, T, f, u_0) h^2, \tag{5.11}$$

where $C_2(\alpha, \Omega_1, T, f, u_0)$ *is the positive constant dependent of* α, Ω_1, T, f, *and* u_0.

PROOF. We set $e = P_h u - u_h$, $\eta = \rho_h p - p_h$, $\mu = s_h \lambda(\gamma_0 u, u_0) - \lambda_h(\gamma_0 u_h, u_0)$. Then problems (Q) and (Q$_h$) yield

$$\begin{aligned} \left(\frac{\partial e}{\partial t}, v\right) &+ a_0(e,v) + a_1(u;u,v) - a_1(u_h;u_h,v) + a_1(w_0;e,v) + a_1(e;\psi,v) \\ &+ a_1(\psi;e,v) + \langle \gamma_0 v, \lambda_h(\gamma_0 P_h u, u_0) - \lambda_h(\gamma_0 u_h, u_0)\rangle - (\eta, \operatorname{div} v) \\ &+ a_0(u - P_h u, v) + \langle \gamma_0 v, \lambda(\gamma_0 u, u_0) - \lambda_h(\gamma_0 P_h u, u_0)\rangle \\ &+ a_1(w_0; u - P_h u, v) + a_1(u - P_h; \psi, v) + a_1(\psi; u - P_h u, v) \\ &- (p - \rho_h p, \operatorname{div} v) = 0, \qquad \forall\, v \in W_h, \end{aligned} \tag{5.12}$$

$$(q, \operatorname{div}(u - u_h)) = 0, \qquad \forall\, q \in \bar{M}_h. \tag{5.13}$$

$$b(t,\mu,\mu') - \frac{1}{2}\langle \gamma_0(u - u_h), \mu'\rangle + \langle G(t, \gamma_0(u - u_h)), \mu'\rangle = 0, \qquad \forall\, \mu' \in \Lambda_h. \tag{5.14}$$

According to Lemma 4.1, $\lambda_h(\gamma_0 P_h u, u_0) - \lambda_h(\gamma_0 u_h, u_0) = \lambda_h(e, 0) \in L^2(0,T;\Lambda_h)$ satisfies

$$\|\lambda_h(\gamma_0 e, 0)\|_{L^2(0,T;\Lambda)} \le c_7 \|e\|_{L^2(0,T;W_{0h})}, \tag{5.15}$$

$$\int_0^t [a_1(w_0;e,e) + \langle \gamma_0 e, \lambda_h(\gamma_0 e, 0)\rangle] \, d\tau \ge 0, \qquad \forall\, 0 \le t \le T. \tag{5.16}$$

Hence, taking $v = e$ in (5.12), $q = \eta$ in (5.13), and $\mu' = \mu$ in (5.14), we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|e|_0^2 &+ a_0(e,e) + a_1(u;u,e) - a_1(u_h;u_h,e) + a_1(w_0;e,e) + a_1(e;\psi,e) \\ &+ a_1(\psi;e,e) + \langle \gamma_0 e, \lambda_h(\gamma_0 e, 0)\rangle + a_0(u - P_h u, e) - (p - \rho_h p, \operatorname{div} e) \\ &+ a_1(w_0; u - P_h u, e) + a_1(\psi; u - P_h, e) + a_1(u - p_h u; \psi, e) \\ &+ \langle \gamma_0 e, \lambda(\gamma_0 u, u_0) - \lambda_h(\gamma_0 P_h u, u_0)\rangle = 0, \end{aligned} \tag{5.17}$$

$$\begin{aligned} b(t,\mu,\mu) &- \frac{1}{2}\langle \gamma_0(u - u_h), \mu\rangle + \langle G(t, \gamma_0(u - u_h, 0)), \mu\rangle \\ &+ b(t, \lambda(\gamma_0 u, u_0) - s_h \lambda(\gamma_0 u, u_0), \mu) = 0. \end{aligned} \tag{5.18}$$

Using the previous assumptions, we have

$$\begin{aligned} a_0(e,e) &\ge \alpha |e|_1^2, \\ |a_1(u;u,e) - a_1(u_h;u_h,e)| &\le |a_1(e,u,e)| + |a_1(u_h;e,e)| + |a_1(u - P_h u; u, e)|, \\ |a_1(e;u,e)| &\le c_0 |e|_0 |e|_1 |u|_1 \le \frac{\alpha}{16}|e|_1^2 + \frac{4}{\alpha} c_0^2 |u|_1^2 |e|_0^2, \\ |a_1(u_h;e,e)| &\le c_0 |e|_1^{3/2} |e|_0^{1/2} |u_h|_0^{1/2} |u_h|_1^{1/2} \le \frac{\alpha}{16}|e|_1^2 + \left(\frac{8}{\alpha}\right)^3 c_0^4 |u_h|_0^2 |u_h|_1^2 |e|_0^2, \\ |a_1(u - P_h u; u, e)| &\le \frac{\alpha}{16}|e|_1^2 + \alpha^{-1} c_0^2 |u - P_h u|_0^2 |u|_1^2 + \frac{\alpha}{16}|u - P_h u|_1^2 + \frac{4}{\alpha} c_0^2 |u|_1^2 |e|_0^2, \\ |a_1(e;\psi,e) + a_1(\psi;e,e)| &\le \frac{\alpha}{10}|e|_1^2, \end{aligned}$$

$$|a_1(w_0;u-P_hu,e)+a_1(u-P_hu;\psi,e)+a_1(\psi;u-P_hu,e)| \\ \leq \frac{\alpha}{10}|e|_1^2+\frac{\alpha}{8}|u-P_hu|_1^2+\frac{4}{\alpha}|w_0|^2|e|_0^2,$$

$$|a_0(u-P_hu;e)|\leq 36\alpha|u-P_hu|_1^2+\frac{\alpha}{16}|e|_1^2,$$
$$|(p-\rho_hp,\operatorname{div}e)|\leq\frac{\alpha}{16}|e|_1^2+\frac{8}{\alpha}|p-\rho_hp|_0^2.$$

Moreover, by the trace theorem, there holds

$$|\langle\gamma_0e,\lambda(\gamma_0u,u_0)-\lambda_h(\gamma_0P_hu,u_0)\rangle|\leq\|e\|_{H^{1/2}(\Gamma_2)^2}\|\lambda(\gamma_0u,u_0)-\lambda_h(\gamma_0P_hu,u_0)\|_\Lambda \\ \leq\frac{\alpha}{16}|e|_1^2+c_8\|\lambda(\gamma_0u,u_0)-\lambda_h(\gamma_0P_hu,u_0)\|_\Lambda^2.$$

Combining (5.17),(5.18) with above estimates yields

$$\frac{d}{dt}|e|_0^2+\alpha|e|_1^2+\langle\gamma_0e,\lambda(\gamma_0e,0)\rangle+a_1(w_0;e,e) \\ \leq c(\alpha,\Omega_1)\left(|u-P_hu|_1^2+|p-\rho_hp|_0^2+\|\lambda(\gamma_0u,u_0)-\lambda_h(\gamma_0P_hu,u_0)\|_\Lambda^2\right)+g(t)|e|_0^2, \tag{5.19}$$

where

$$g(t)=\frac{16}{\alpha}c_0^2|u|_1^2+2\left(\frac{4}{\alpha}\right)^3c_0^4|u_h|_0^2|u_h|_1^2+\frac{4}{\alpha}|w_0|^2\in L^1(0,T).$$

Integrating (5.19) and using (5.16), we obtain

$$|e(t)|_0^2+\alpha\int_0^t|e|_1^2\,d\tau\leq c(\alpha,\Omega_1,T,f,u_0)\left\{\int_0^T|u-P_hu|_1^2\,d\tau\right. \\ \left.+\int_0^T|p-\rho_hp|_0^2\,d\tau+\int_0^T\|\lambda(\gamma_0u,u_0)-\lambda_h(\gamma_0P_hu,u_0)\|_\Lambda^2\,d\tau\right\}. \tag{5.20}$$

Thanks to Lemmas 2.1 and 4.1, one has

$$\|\lambda(\gamma_0u,u_0)-\lambda(\gamma_0P_hu,u_0)\|_{L^2(0,T;\Lambda)}\leq c_3\|u-P_hu\|_{L^2(0,T;W)}, \tag{5.21}$$

$$b\left(t,s_h\lambda\left(\gamma_0P_hu,u_0\right)-\lambda_h\left(\gamma_0P_hu,u_0\right),\mu'\right)=b(t,s_h\lambda(\gamma_0P_hu,u_0) \\ -\lambda(\gamma_0P_hu,u_0),\mu'),\qquad\forall\,\mu'\in\Lambda_h. \tag{5.22}$$

Using (2.14),(2.15), we obtain from (5.22) that

$$\|s_h\lambda(\gamma_0P_hu,u_0)-\lambda_h(\gamma_0P_hu,u_0)\|_{L^2(0,T:\Lambda)}\leq c_2c_1^{-1}\|s_h\lambda(\gamma_0P_hu,u_0)-\lambda(\gamma_0P_hu,u_0)\|_{L^2(0,T:\Lambda)}. \tag{5.23}$$

Hence, (5.23) and (H_2) yield

$$\|\lambda(\gamma_0P_hu,u_0)-\lambda_h(\gamma_0P_hu,u_0)\|_{L^2(0,T;\Lambda)}^2 \\ \leq\left(2+2c_2^2c_1^{-2}\right)\|s_h\lambda(\gamma_0P_hu,u_0)-\lambda(\gamma_0P_hu,u_0)\|_{L^2(0,T:\Lambda)}^2 \\ \leq c_9h^2\int_0^T\|\lambda(\gamma_0P_hu,u_0)\|_{H^{1/2}(\Gamma_2)^2}^2\,d\tau. \tag{5.24}$$

Therefore, (5.23),(5.24) and Lemma 2.1 yield

$$\|\lambda(\gamma_0u,u_0)-\lambda_h(\gamma_0P_hu,u_0)\|_{L^2(0,T;\Lambda)}^2 \\ \leq 2\|\lambda(\gamma_0u,u_0)-\lambda(\gamma_0P_hu,u_0)\|_{L^2(0,T;\Lambda)}^2+2\|\lambda(\gamma_0P_hu,u_0)-\lambda_h(\gamma_0P_hu,u_0)\|_{L^2(0,T;\Lambda)}^2 \\ \leq 2c_3^2\|u-P_hu\|_{L^2(0,T;W)}^2+2c_9h^2\int_0^T\|\lambda(\gamma_0P_hu,u_0)\|_{H^{1/2}(\Gamma_2)^2}^2\,d\tau. \tag{5.25}$$

Combining (5.20) and (4.6)–(4.8) yields

$$|e(t)|_0^2 + \alpha \int_0^t |e|_1^2 \, d\tau \le C(\alpha, \Omega_1, T, f, u_0) h^2. \tag{5.26}$$

Using again the triangle inequality, one has

$$|u(t) - u_h(t)|_0^2 + \alpha \int_0^t |u - u_h|_1^2 \, d\tau \le C(\alpha, \Omega_1, T, f, u_0) h^2. \tag{5.27}$$

Recalling [3], we have

$$\begin{aligned} &\frac{1}{2} \int_0^T \langle \gamma_0(u - u_h), \mu \rangle \, dt - \int_0^T \langle G(t, \gamma_0(u - u_h)), \mu \rangle \, dt \\ &\quad \le C(\alpha, \Omega_1, T, f, u_0) \|u - u_h\|_{L^2(0,T;W)} \|\mu\|_{L^2(0,T;\Lambda)}. \end{aligned} \tag{5.28}$$

Combining (5.18) and (5.28) and using (2.14),(2.15), one has

$$\begin{aligned} \|\mu\|_{L^2(0,T;\Lambda)} &\le C(\alpha, \Omega_1, T, f, u_0) \|u - u_h\|_{L^2(0,T;W)} \\ &\qquad + C(\alpha, \Omega_1, T, f, u_0) \|\lambda(\gamma_0 u, u_0) - s_h \lambda(\gamma_0 u, u_0)\|_{L^2(0,T;\Lambda)} \\ &\le C(\alpha, \Omega_1, T, f, u_0) h. \end{aligned} \tag{5.29}$$

Applying the triangle inequality, we derive from (5.29) that

$$\|\lambda(\gamma_0 u, u_0) - \lambda_h(\gamma_0 u_h, u_0)\|_{L^2(0,T;\Lambda)} \le C(\alpha, \Omega_1, T, f, u_0) h. \tag{5.30}$$

Therefore, (5.27) and (5.30) have completed the proof of Theorem 5.2. ∎

REMARK. According to Theorems 5.1 and 5.2, we have

$$|\bar{u}(t) - u_h(t)|_0^2 + \alpha \int_0^t |\bar{u} - u_h|_1^2 \, d\tau \le C_1(\alpha, \Omega_1, T, f, u_0) h^2 + C_2(\alpha, T, f, u_0) \varepsilon^2 \int_0^T \delta^2(t) \, dt. \tag{5.31}$$

Now, we aim to derive the error estimates of the pressure. Applying Green's formula to problem (S) in Ω_1, we obtain

$$\begin{aligned} &\left(\frac{\partial \bar{u}}{\partial t}, v \right) + a_0(\bar{u}, v) + a_1(\bar{u}; \bar{u}, v) + a_1(w_0; \bar{u}, u) + a_1(\psi; \bar{u}, v) \\ &\qquad + a_1(\bar{u}; \psi, v) - (\bar{p}, \operatorname{div} v) = (f, v), \qquad \forall v \in X_{0h}^2. \end{aligned} \tag{5.32}$$

We set $E = \bar{u} - u_h, \eta = \rho_h \bar{p} - p_h$, then (5.32) and ($Q_h$) imply

$$\begin{aligned} &\left(\frac{\partial E}{\partial t}, v \right) + a_0(E, v) + a_1(E; \bar{u}, v) - a_1(u_h; E, v) - (\bar{p} - \rho_h \bar{p}, \operatorname{div} v) \\ &+ a_1(w_0; E, v) + a_1(\psi; E, v) + a_1(E; \psi, v) = (\eta, \operatorname{div} v), \qquad \forall v \in X_{0h}^2. \end{aligned} \tag{5.33}$$

Integrating (5.33), we obtain

$$\begin{aligned} &\left(\int_0^T \eta \, dt, \operatorname{div} v \right) = (E(T) - E(0), v) - \int_0^T (\bar{p} - \rho_h \bar{p}, \operatorname{div} v) \, dt \\ &+ \int_0^T [a_1(w_0; E, v) + a_1(\psi; E, v) + a_1(E; \psi, v)] \, dt + \int_0^T [a_0(E, v) + a_1(E; \bar{u}, v) + a_1(u_h; E, v)] \, dt. \end{aligned} \tag{5.34}$$

Applying Poincaré's inequality, (3.30) and (2.12)–(5.13), one has

$$|(E(T)-E(0),v)| \le 2\gamma(\Omega_1)\sup_{0\le t\le T}|E(t)|_0|v|_1,$$

$$\left|\int_0^T(\bar p-\rho_h\bar p,\operatorname{div}v)\,dt\right| \le \sqrt{2T}\left(\int_0^T|\bar p-\rho_h\bar p|_0^2\,dt\right)^{1/2}|v|_1,$$

$$\left|\int_0^T a_0(E,v)\,dt\right| \le 3\alpha\sqrt{T}\left(\int_0^T|E|_1^2\,dt\right)^{1/2}|v|_1,$$

$$\left|\int_0^T[a_1(E;\bar u,v)+a_1(u_h;E,v)]\,dt\right|$$
$$\le 2c_0\gamma(\Omega_1)\left(\int_0^T\left(|\bar u|_1^2+|u_h|_1^2\right)dt\right)^{1/2}\left(\int_0^T|E|_1^2\,dt\right)^{1/2}|v|_1,$$

$$\int_0^T[a_1(w_0;E,v)+a_1(\psi;E,v)+a_1(E;\psi,v)]\,dt \le \left(\gamma(\Omega_1)|w_0|+\frac{\alpha}{4}\right)\sqrt{T}\left(\int_0^T|E|_1^2\right)^{1/2}|v|_1.$$

Combining (5.34) and above estimates yields

$$\begin{aligned}\frac{\left(\int_0^T\eta\,dt,\operatorname{div}v\right)}{|v|_1} &\le \gamma(\Omega_1)c(\alpha,T,f,u_0)\\ &\times\left\{\sup_{0\le t\le T}|E(t)|_0+\left(\int_0^T|E|_1^2\,dt\right)^{1/2}+\left(\int_0^T|\bar p-\rho_h\bar p|_0^2\,dt\right)^{1/2}\right\}.\end{aligned}\tag{5.35}$$

Applying (5.31) and (H_3),(H_4), we obtain

$$\begin{aligned}\left|\int_0^T\eta\,dt\right|_0 &\le \beta_*^{-1}\sup_{v\in X_{0h}^2}\frac{\left(\int_0^T\eta\,dt,\operatorname{div}v\right)}{|v|_1}\\ &\le C_3(\alpha,\Omega_1,T,f,u_0)\varepsilon\left(\int_0^T\delta^2(t)\,dt\right)^{1/2}+C_4(\alpha,\Omega_1,T,f,u_0)\,h.\end{aligned}\tag{5.36}$$

Using again the triangle inequality and (5.36), we obtain the error estimates of the pressure.

THEOREM 5.3. *Under the assumptions of Theorem 5.2, $p_h(t)$ satisfies the following approximate accuracy:*

$$\left|\int_0^T(\bar p-p_h)\,dt\right|_0^2 \le C_5(\alpha,\Omega_1,T,f,u_0)\varepsilon^2\int_0^T\delta^2(t)\,dt+C_6(\alpha,\Omega_1,T,f,u_0)h^2.\tag{5.37}$$ ∎

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